1.7 THE RELATIONSHIP BETWEEN STRESS AND RATE OF STRAIN

We will now split the stress tensor into isotropic and deviatoric parts by writing

$$\sigma_{ij} = -p\,\delta_{ij} + d_{ij}$$

Since we know that σ_{ij} is symmetric it follows that d_{ij} must be also.

To gain some insight in how to choose d_{ij} we consider a 2D example (see diagram):
Consider the line elements AB and AC to be initially perpendicular.
The angular velocity or rotation of AB relative to A is $\partial u_2/\partial x_1$.
The rotation of AC relative to A is $-\partial u_1/\partial x_2$. Therefore the average rotation is $\frac{1}{2}(\frac{\partial u_2}{\partial x_1}-\frac{\partial u_1}{\partial x_2})$.

The stretching per unit length of AB relative to A is $\partial u_1/\partial x_1$ and that of AC relative to A is $\partial u_2/\partial x_2$.

Therefore the average stretching per unit length is $\frac{1}{2}(\frac{\partial u_1}{\partial x_1}+\frac{\partial u_2}{\partial x_2})$.

In general, performing a 3D Taylor expansion about a point we have:

$$\underline{u}(\underline{x}+\delta\underline{x}) = \underline{u}(\underline{x})+(\delta\underline{x}.\underline{\nabla})\underline{u}+O(\delta\underline{x})^2$$

so that to order $\delta\underline{x}$:

$$\delta u_i = \left[\underline{u}(\underline{x}+\delta\underline{x})-\underline{u}(\underline{x})\right]_i = \delta x_j \partial u_i/\partial x_j \qquad \text{(sc)}$$

$$\equiv \delta x_j\left[\frac{1}{2}(\frac{\partial u_i}{\partial x_j}+\frac{\partial u_j}{\partial x_i}) + \frac{1}{2}(\frac{\partial u_i}{\partial x_j}-\frac{\partial u_j}{\partial x_i})\right]$$

The first term represents stretching of a fluid element so we define the rate of strain tensor e_{ij} by:

$$e_{ij} = \frac{1}{2}(\frac{\partial u_i}{\partial x_j}+\frac{\partial u_j}{\partial x_i})$$

Note that e_{ij} is symmetric. The second term above represents rotation without deformation with angular velocity $\frac{1}{2}$curl $\underline{u}$. The quantity curl $\underline{u}$ is the vorticity of the fluid. If the vorticity is zero the fluid is known as irrotational. Incompressible, irrotational flows were dealt with in the 2nd year fluids course (the governing equation is Laplace's equation for the velocity potential). A flow without vorticity is necesssarily inviscid, but inviscid flows may possess vorticity.

We now wish to postulate a form for the deviatoric part of the stress tensor. From above we see that deformation of fluid elements only occurs through the rate of strain tensor e_{ij}. We shall follow the ideas of Stokes and assume that the fluid has the following properties:

(i) The relation between stress and rate of strain is independent of the rigid body rotation of a fluid element, in other words:

$$d_{ij} = \text{some function of } e_{ij}$$

(ii) The fluid is homogeneous: the properties of the fluid do not vary with position. This means that d_{ij} does not depend explicitly on $\underline{x}$.

(iii) The fluid is isotropic: there is no preferred direction that the fluid 'wants' to travel in.

(iv) In the absence of motion, the stress is hydrostatic. In other words, when e_{ij} is zero we have d_{ij} zero also, so that $\sigma_{ij} = -p\delta_{ij}$. We call $-p$ the pressure at a point, and it is the mean of the three normal stresses at that point (see diagram) if there is no motion.

These four properties define a **Stokesian fluid.** Experimentally, it has been shown that a large class of fluids appear to be Stokesian.
In addition, for the purposes of this course we make the following further assumption:

(v) We assume the fluid to be **Newtonian**: this means that there is a linear relationship between stress and rate of strain. This assumption has been validated experimentally under a wide range of conditions.

Assumptions (i), (iv) and (v) together imply that

$$d_{ij} = A_{ijkm} e_{km} \tag{sc}$$

where A is some tensor.
Assumption (iii) implies that A is isotropic which means that it can be written in the form

$$A_{ijkm} = \mu\delta_{ik}\delta_{jm} + \mu'\delta_{im}\delta_{jk} + \lambda\delta_{ij}\delta_{km}$$

(the most general form for a fourth order isotropic tensor) where μ, μ' and λ are all scalar quantities (see M2M1)† and are independent of $\underline{x}$ in view of assumption (ii). Now, since d_{ij} is symmetric we must have $A_{ijkm} = A_{jikm}$, and this implies $\mu' = \mu$. So d_{ij} can be written in the form

$$
\begin{aligned}
d_{ij} &= \mu(\delta_{ik}\delta_{jm}+\delta_{im}\delta_{jk})\, e_{km} + \lambda\delta_{ij}\delta_{km}e_{km} \\
&= \mu\, e_{ij} + \mu\, e_{ji} + \lambda\delta_{ij}e_{kk} \\
&= 2\mu\, e_{ij} + \lambda\delta_{ij}e_{kk} \quad (\text{since } e_{ij}=e_{ji}) \qquad \text{(sc)}
\end{aligned}
$$

Thus we have $\qquad \sigma_{ij} = -p\delta_{ij} + 2\mu e_{ij} + \lambda\delta_{ij}e_{kk}.$

If we denote by $-\hat{p}$ (hydrostatic pressure) the mean of the normal stresses, i.e.

$$-\hat{p} \equiv \tfrac{1}{3}(\sigma_{11}+\sigma_{22}+\sigma_{33}),$$

then we have (putting $i=j$ above):

$$-3\hat{p} = -3p + (2\mu+3\lambda)e_{kk} \qquad \text{(sc)}$$

$$\Rightarrow \qquad p-\hat{p} = (\tfrac{2}{3}\mu+\lambda)\text{div }\underline{u}$$

$$\Rightarrow \text{(by continuity eqn)} \qquad p-\hat{p} = -(\tfrac{2}{3}\mu+\lambda)\frac{1}{\rho}\frac{d\rho}{dt}.$$

The term in brackets is known as the coefficient of <u>bulk</u> <u>viscosity.</u> For compressible fluids, p is taken to be the thermodynamic pressure which can be found from the equation of state. It is common to take this equal to the hydrostatic pressure $\hat{p}$, in which case we must have

$$\lambda = -2\mu/3$$

(For incompressible fluids, $e_{kk}=0$, so that λ does not appear in our expression for σ_{ij} so that this final assumption is not required).

The full expression for the stress tensor is now

$$\boxed{\sigma_{ij} = -p\,\delta_{ij} + 2\mu(e_{ij}-\tfrac{1}{3}\delta_{ij}e_{kk})} \qquad \text{(sc)}$$

Note that $e_{kk} = e_{11}+e_{22}+e_{33} =$ div $\underline{u}$. The constant scalar μ is known as <u>**the coefficient of viscosity**</u> of the fluid. Typical values are 0.0002 g/cm/sec (air), 0.01 (water), 23.3 (glycerine).

† or the book by Aris (see recommended reading)

We can now substitute our form for the stress tensor into Cauchy's equation of motion:

$$\rho\frac{du_i}{dt} = \rho g_i + \frac{\partial}{\partial x_j}\left[-\,p\,\delta_{ij} + 2\mu(e_{ij} - \tfrac{1}{3}\delta_{ij}e_{kk})\right]$$

$$= \rho g_i - \frac{\partial p}{\partial x_i} + 2\mu\frac{\partial}{\partial x_j}\left[(e_{ij} - \tfrac{1}{3}\delta_{ij}e_{kk})\right] \qquad \text{(sc)}$$

since the coefficient of viscosity μ is independent of $\underline{x}$. Analysing the terms in the square bracket we see that:

$$\frac{\partial\, e_{ij}}{\partial x_j} = \frac{1}{2}\frac{\partial}{\partial x_j}\left[\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right] = \frac{1}{2}\left[\frac{\partial^2 u_i}{\partial {x_j}^2} + \frac{\partial}{\partial x_i}\left(\frac{\partial u_j}{\partial x_j}\right)\right] \qquad \text{(sc over j)}$$

$$= \tfrac{1}{2}\left[\nabla^2\underline{u} + \underline{\nabla}(\text{div } \underline{u})\right]_i$$

and

$$\frac{\partial}{\partial x_j}(\delta_{ij}e_{kk}) = \frac{\partial}{\partial x_i}(e_{kk}) = \frac{\partial}{\partial x_i}\left(\frac{\partial u_k}{\partial x_k}\right) = \left[\underline{\nabla}(\text{div } \underline{u})\right]_i$$

Substituting these expressions into the equation of motion we have

$$\rho\frac{du_i}{dt} = \rho g_i - [\underline{\nabla}p]_i + \mu\left[\nabla^2\underline{u} + \tfrac{1}{3}\underline{\nabla}(\text{div } \underline{u})\right]_i$$

Writing in terms of vectors:

$$\boxed{\frac{d\underline{u}}{dt} = \underline{g} - \frac{1}{\rho}\underline{\nabla}p + \nu\left(\nabla^2\underline{u} + \tfrac{1}{3}\underline{\nabla}(\text{div } \underline{u})\right) \text{ NAVIER-STOKES EQN OF MOTION}}$$

where we have defined $\nu = \mu/\rho \equiv$ **the kinematic viscosity of the fluid.**